Potential scattering in constant magnetic field: Spectral asymptotics and Levinson formula

1995 J. Phys. A: Math. Gen. 28 3493

(http://iopscience.iop.org/0305-4470/28/12/021)

J. Phys. A: Math. Gen. 28 (1995) 3493–3509. Printed in the UK

Potential scattering in constant magnetic field: spectral asymptotics and Levinson formula

Vadim V Kostrykin†‖, Andrei A Kvitsinsky‡‖ and Stanislav P Merkuriev§¶

† Institüt für Reine und Angewandte Mathematik, RWTH Aachen, Templegraben 55, D-52056 Aachen, Germany

‡ Physics Department, California State University, Long Beach, CA 90840, USA

§ Department of Mathematical and Computational Physics, Institute for Physics, University of St Petersburg, 198904 St Petersburg, Russia

Received 10 March 1995

Abstract. The scattering problem related to the Schrödinger operator with an external constant homogeneous magnetic field is considered. The behaviour of the corresponding S-matrix and resolvent operator near the Landau thresholds is studied. The large-energy asymptotics of the resolvent operator are evaluated. A Levinson-type formula is obtained. It relates the number of bound states to the determinant of the scattering matrix in the scattering threshold.

1. Introduction

The Schrödinger operator with an external constant homogeneous magnetic field

$$H = -(\nabla - \mathrm{i}a(x))^2 + V(x) \qquad a(x) = \tfrac{1}{2}B \times x \tag{1.1}$$

provides a problem which has as much mathematical interest as importance due to applications in astrophysics and solid-state physics (see the review [1] and references therein). The basic mathematical aspects of the scattering problem for this operator have been studied in [2] where the existence and completeness of the corresponding wave operators were proved for a large class of potentials $V(x)$.

This paper is concerned with problems arising in context of near-threshold scattering for the operator (1.1), when energy approaches one of the Landau levels [3] determining branches of continuous spectrum. Related problems, such as the threshold structure of the resolvent operator and S-matrix, as well as spectral identities (sum rules) like the Levinson formula, are of much interest in potential scattering theory [4]. A lot of work has been done in this field concerning Schrödinger operators without external fields. For instance, quite complete results were obtained on low-energy behaviour of S-matrix for various types of potentials including slowly decreasing ones (see [4] and recent works [5–9]) and a number of generalizations of the Levinson formula [10] were derived. Among them are complete series of spectral identities for radial [11] and three-dimensional [12] Schrödinger operators, a two-dimensional analogue [13] of the Levinson formula, its generalizations for slowly decreasing [14–17], non-local [18], non-central [19] and periodic [20] potentials. Similar

‖ Permanent address: Department of Mathematical and Computational Physics, Institute for Physics, University of St Petersburg, 198904 St Petersburg, Russia.

¶ Deceased.

0305-4470/95/123493+17$19.50

problems have also been studied for three-body systems [21–25]. Witten index theorems in supersymmetric quantum mechanics [26–28] also present extended versions of the Levinson formula.

However, such problems have not been studied for the operator (1.1) and this is the goal of our work. We restrict ourselves to the case where the electrostatic potential $V(x)$ is azimuthally symmetric, so that the problem is actually two-dimensional. Also, the potential is supposed to decrease fast enough as $|x| \to \infty$ (roughly speaking, faster than $|x|^{-3}$).

The main results of the paper consist of studying the threshold behaviour of the S-matrix (theorem 2.5), evaluating the near-threshold and high-energy asymptotics of the resolvent operator (theorem 2.7), and deriving an analogue of the Levinson formula (theorem 2.8). To make the structure of the paper more transparent, we describe these and other important results in section 2, leaving their proofs to subsequent sections.

2. Main results

The Hamiltonian (1.1) of a particle moving in the external magnetic field $\boldsymbol{B} = (0, 0, B)$, $B > 0$ is of the form

$$H = H_0 + V \qquad H_0 = -\Delta_{x_\perp} - \partial_{x_3}^2 + \frac{B^2}{4}x_\perp^2 + Bl_3$$

where $x_\perp = (x_1, x_2)$ and l_3 is projection of the angular momentum onto the direction of the field $\boldsymbol{B}$. We suppose that the potential V is azimuthally symmetric, i.e. $V = V(|x_\perp|, x_3) = V(\rho, x_3)$ with $\rho = |x_\perp|$.

In this case, the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ can be decomposed into a direct sum of orthogonal subspaces $\mathcal{H}_m = L^2(\mathbb{R}_+^2; \rho\mathrm{d}\rho\mathrm{d}x_3)$ corresponding to fixed values m of the projection l_3. The corresponding Hamiltonians with fixed m are of the form

$$H_m = H_{0m} + V \qquad H_{0m} = h_m^{\mathrm{osc}} + Bm - \partial_{x_3}^2 \tag{2.1}$$

$$h_m^{\mathrm{osc}} = -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2} + \frac{B^2}{4}\rho^2. \tag{2.2}$$

The spectrum of H_{0m} is absolutely continuous and consists of an infinite number of branches $[\epsilon_{mp}, \infty)$. Their thresholds are the Landau levels

$$\epsilon_{mp} = B(m + |m| + 1 + 2p) \qquad p \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

We denote by $\Gamma_m = \{\epsilon_{mp}\}_{p=0}^\infty$ the set of all thresholds.

We assume the potential V to be a smooth function satisfying the conditions

$$\begin{gathered} |V(\rho, x_3)| \leqslant C_0 \left[\frac{\rho^2 + x_3^2}{1 + \rho^2 + x_3^2}\right]^{-1+\mu_0} \qquad 0 < \mu_0 \leqslant 1 \\ \int_{\mathbb{R}_+^2} \rho\mathrm{d}\rho\, \mathrm{d}x_3\, (1 + |x_3|)^{2l}|V(\rho, x_3)| < \infty \end{gathered} \tag{2.3}$$

with $l \geqslant 2$. Under this assumption H_m is self-adjoint on $\mathcal{D}(H_{0m})$ and has a finite discrete spectrum [2].

The existence and completeness of the wave operators

$$U_m^{(\pm)} = s - \lim_{t \to \mp\infty} \mathrm{e}^{\mathrm{i}H_m t}\mathrm{e}^{-\mathrm{i}H_{0m} t}$$

were proved [2] for a large class of potentials including those of the type (2.3).

As the scattering operator $S_m = U_m^{(-)*}U_m^{(+)}$ commutes with H_{0m}, it can be represented as a direct integral of the S-matrices at fixed energy:

$$S_m = \int_{\epsilon_{m0}}^{\infty} \oplus S_m(E)\,\mathrm{d}E.$$

Each $S_m(E)$ acts in an accessory space $\hat{\mathcal{H}}_m(E)$ related to a direct integral decomposition of the Hilbert space $\mathcal{H}_m$ with respect to H_{0m}

$$\mathcal{H}_m \sim L^2\left([\epsilon_{m0},\infty),\mathrm{d}E;\hat{\mathcal{H}}_m(E)\right) = \int_{\epsilon_{m0}}^{\infty} \oplus\hat{\mathcal{H}}_m(E) \tag{2.4}$$

where H_{0m} acts in $\hat{\mathcal{H}}_m(E)$ as multiplication by E.

Let us describe the structure of $\hat{\mathcal{H}}_m(E)$. Consider a unitary transform $\mathcal{F}_m : \mathcal{H}_m \to \mathcal{H}_m$ provided by the eigenfunction expansion associated with H_{0m}:

$$(\mathcal{F}_m f)(p,k_3) = \mathrm{l.i.m.}\int_{\mathbb{R}_+^2} \rho\mathrm{d}\rho\,\mathrm{d}x_3\;\mathcal{F}_m(p,k_3;\rho,x_3)f(\rho,x_3) \qquad k_3\in\mathbb{R} \quad p\in\mathbb{N}_0 \tag{2.5}$$

with the kernel

$$\mathcal{F}_m(p,k_3;\rho,x_3) = \frac{\mathrm{e}^{-\mathrm{i}k_3x_3}}{\sqrt{2\pi}}\phi_{mp}(\rho) \tag{2.6}$$

where ϕ_{mp} stands for the eigenfunction of the Hamiltonian (2.2) that is expressed in terms of the Laguerre polynomials [30],

$$\begin{aligned}&\phi_{mp}(\rho) = c_{mp}\mathrm{e}^{-r/2}r^{|m|/2}L_p^{|m|}(r)\\ &r = \frac{B}{2}\rho^2 \qquad c_{mp}^2 = \frac{Bp!}{(p+|m|)!}.\end{aligned} \tag{2.7}$$

The transform (2.5) generates the diagonal representation of H_{0m}: $\mathcal{F}_m H_{0m}\mathcal{F}_m{}^* = \epsilon_{mp} + k_3^2$.

Let $\hat{f} = \mathcal{F}_m f$, $f\in\mathcal{H}_m$. The inner product in $\mathcal{H}_m$ can be written as

$$\begin{aligned}(f,g) &= \int_{-\infty}^{+\infty}\mathrm{d}k_3\sum_{p=0}^{\infty}\overline{\hat{f}(p,k_3)}\hat{g}(p,k_3)\\ &= \int_{\epsilon_{m0}}^{\infty}\mathrm{d}E\sum_{p=0}^{n(E)}\nu_{mp}(E)\sum_{\pm}\overline{\hat{f}(p,\pm k_{mp}(E))}\hat{g}(p,\pm k_{mp}(E)).\end{aligned} \tag{2.8}$$

Here $k_{mp}(E) = \sqrt{E-\epsilon_{mp}}$, $\nu_{mp}(E) = \left(2k_{mp}(E)\right)^{-1}$ and $n(E)$ is the largest integer satisfying $B(m+|m|+1+2n(E)) \leqslant E$. In other words, $n(E)$ yields the number of Landau thresholds open at the energy E. From equation (2.8) it follows (see, e.g., [31]) that the layer $\hat{\mathcal{H}}_m(E)$ of (2.4) is given by direct sum of a finite number of two-dimensional linear spaces $\mathbb{C}^2$ with elements

$$\hat{\mathcal{H}}_m(E) \ni g = \bigoplus_{p=0}^{n(E)}\nu_{mp}(E)g_p \qquad g_p = \begin{pmatrix} g_p^{(+)} \\ g_p^{(-)}\end{pmatrix}. \tag{2.9}$$

This is the space where the S-matrix $S_m(E)$ acts. To evaluate its explicit form it is convenient to introduce a mapping $F_V(E) : \mathcal{H}_m \to \hat{\mathcal{H}}_m(E)$ for $E\in[\epsilon_{m0},\infty)\setminus\Gamma_m$

$$\begin{aligned}(F_V(E)f)_p = &\sqrt{\nu_{mp}(E)}\int_{\mathbb{R}_+^2}\rho\mathrm{d}\rho\,\mathrm{d}x_3\;V(\rho,x_3)f(\rho,x_3)\\ &\times\left\{\mathcal{F}_m(p,k_{mp}(E);\rho,x_3)\begin{pmatrix}1\\0\end{pmatrix} + \mathcal{F}_m(p,-k_{mp}(E);\rho,x_3)\begin{pmatrix}0\\1\end{pmatrix}\right\}.\end{aligned} \tag{2.10}$$

The adjoint operator $F_V(E)^* : \hat{\mathcal{H}}_m(E) \to \mathcal{H}_m$ is given by

$$\left(F_V(E)^* g\right)(\rho, x_3) = V(\rho, x_3) \sum_{p=0}^{n(E)} \sqrt{v_{mp}(E)}$$
$$\times \left\{\mathcal{F}_m(p, -k_{mp}(E); \rho, x_3) g_p^{(+)} + \mathcal{F}_m(p, k_{mp}(E); \rho, x_3) g_p^{(-)}\right\}. \tag{2.11}$$

The S-matrix can be expressed in terms of these mappings and the operator

$$A_m(z) = |V|^{1/2} R_{0m}(z) V^{1/2} \tag{2.12}$$

where $V^{1/2} = V|V|^{-1/2}$, $R_{0m}(z) = (H_{0m} - z)^{-1}$.

Theorem 2.3. For all $E \in (\epsilon_{m0}, \infty) \setminus \Gamma_m$ the S-matrix is unitary operator in $\hat{\mathcal{H}}_m(E)$ given by

$$S_m(E) = I - 2\pi \mathrm{i} F_{V^{1/2}}(E) \left[I + A_m(E + \mathrm{i}0)\right]^{-1} F_{|V|^{1/2}}(E)^*. \tag{2.13}$$

It is continuously differentiable in E on every open interval $(\epsilon_{mp}, \epsilon_{mp+1})$.

The S-matrix can also be expressed in terms of the asymptotics of the scattering wavefunctions of the Hamiltonian (2.1) related to the modified Lippman–Schwinger equation

$$\psi_m^{(\pm)}(\cdot; p, E) = |V|^{1/2} \mathcal{F}_m(p, \pm k_{mp}(E); \cdot) - A_m(E + \mathrm{i}0) \psi_m^{(\pm)}(\cdot; p, E). \tag{2.14}$$

Theorem 2.4. Equation (2.14) has a unique solution from $L^2(\mathbb{R}_+^2)$ for all $E \in (\epsilon_{m0}, \infty) \setminus \Gamma_m$. The functions

$$\varphi_m^{(\pm)}(\cdot; p, E) = \sqrt{2\pi}\left[\mathcal{F}_m(p, \pm k_{mp}(E); \cdot) - R_{0m}(E + \mathrm{i}0) V^{1/2} \psi_m^{(\pm)}(\cdot; p, E)\right]$$

satisfy the Schrödinger equation $(H_m - E)\varphi_m^{(\pm)}(\cdot; p, E) = 0$ in the sense of distributions. Their asymptotics at $|x_3| \to \infty$ are given by

$$\begin{aligned}
\varphi_m^{(\pm)}(\rho, x_3; p, E) &\sim_{x_3 \to \pm\infty} \mathrm{e}^{\pm \mathrm{i} x_3 k_{mp}(E)} \phi_{mp}(\rho) \\
&\quad + \sum_{p'=0}^{n(E)} \left[t_{p'p}^{(\pm)}(E) - \delta_{p'p}\right] \mathrm{e}^{\pm \mathrm{i} x_3 k_{mp'}(E)} \phi_{mp'}(\rho) \\
\varphi_m^{(\pm)}(\rho, x_3; p, E) &\sim_{x_3 \to \mp\infty} \mathrm{e}^{\pm \mathrm{i} x_3 k_{mp}(E)} \phi_{mp}(\rho) \\
&\quad + \sum_{p'=0}^{n(E)} r_{p'p}^{(\pm)}(E) \mathrm{e}^{\mp \mathrm{i} x_3 k_{mp'}(E)} \phi_{mp'}(\rho).
\end{aligned} \tag{2.15}$$

The coefficients $t^{(\pm)}$ and $r^{(\pm)}$ form the S-matrix elements

$$[S_m(E)]_{pp'} = \left(\frac{k_{mp}(E)}{k_{mp'}(E)}\right)^{1/2} \begin{pmatrix} t_{pp'}^{(+)}(E) & r_{pp'}^{(+)}(E) \\ r_{pp'}^{(-)}(E) & t_{pp'}^{(-)}(E) \end{pmatrix}. \tag{2.16}$$

This theorem shows that the scattering problem related to the Hamiltonian (2.1) is effectively a multichannel 1-dimensional scattering problem with the S-matrix composed of transmission and reflection coefficients.

We now proceed to describe the structure of the S-matrix near the thresholds. In order not to bother with inessential details, we adopt two technical assumptions.

Assumption 2.1. For all $p \in \mathbb{N}_0$

$$\int_{\mathbb{R}_+^2} \rho \mathrm{d}\rho \, \mathrm{d}x_3 \, \phi_{mp}^2(\rho) V(\rho, x_3) \neq 0.$$

Assumption 2.2. There exists a norm limit of the operator $[I + A_m(z)]^{-1}$ as $z \to \epsilon_{mp}$.

Both concern the threshold behaviour of the operator $A_m(z)$. For instance, assumption 2.2 means that all thresholds are not exceptional points. To incorporate the cases where these assumptions fail, one can adjust the technique of [7, 8].

Let

$$\begin{aligned} u_{mp}(\rho, x_3) &= V(\rho, x_3)|V(\rho, x_3)|^{-1/2}\phi_{mp}(\rho) \\ w_{mp}(\rho, x_3) &= |V(\rho, x_3)|^{1/2}\phi_{mp}(\rho). \end{aligned} \tag{2.17}$$

Assumption 2.1 is equivalent to $(u_{mp}, w_{mp}) \neq 0$ that enables one to define the oblique projectors

$$\mathcal{P}_{mp} = \frac{(u_{mp}, \cdot)w_{mp}}{(u_{mp}, w_{mp})} \qquad \mathcal{Q}_{mp} = I - \mathcal{P}_{mp}. \tag{2.18}$$

Then the operator $A_m(z)$ (2.12) near a threshold ϵ_{mp} can be written as follows:

$$A_m(z) = \frac{\mathrm{i}}{2}\frac{\mathcal{P}_{mp}}{k_{mp}(z)}(u_{mp}, w_{mp}) + M_{mp}(z) \tag{2.19}$$

where $k^2_{mp}(z) = z - \epsilon_{mp}$, $\operatorname{Im} k_{mp} \geqslant 0$. The first term represents the threshold singularity of the operator, its form being related to assumption 2.1. Let $M^{(0)}_{mp} = M_{mp}(\epsilon_{mp})$ and

$$T^{(0)}_{mp} = \left[I + \mathcal{Q}_{mp}M^{(0)}_{mp}\mathcal{Q}_{mp}\right]^{-1}\mathcal{Q}_{mp}.$$

This operator is well defined due to assumption 2.2.

Theorem 2.5. Let the condition (2.3) hold true with some $l \geqslant 2$. Then in the limit $E \downarrow \epsilon_{mp}$ the transmission and reflection coefficients (2.16) have the asymptotics:

(i) $p \geqslant p'$:

$$t^{(\pm)}_{pp'}(E) = \sum_{j=0}^{l-1}(\mathrm{i}k_{mp}(E))^j t^{(\pm)}_{pp',j} + \mathrm{o}(k^{l-1}_{mp})$$

$$r^{(\pm)}_{pp'}(E) = \sum_{j=0}^{l-1}(\mathrm{i}k_{mp}(E))^j r^{(\pm)}_{pp',j} + \mathrm{o}(k^{l-1}_{mp})$$

with the leading terms

$$\begin{aligned} t^{(\pm)}_{pp',0} &= \delta_{pp'} - (u_{mp}, w_{mp})^{-1}(u_{mp}, [I - M^{(0)}_{mp}T^{(0)}_{mp}]\eta^{(\pm)}_{pp'}w_{mp'}) \\ &\qquad \pm\tfrac{1}{2}(x_3 u_{mp}, T^{(0)}_{mp}\eta^{(\pm)}_{pp'}w_{mp'}) \\ r^{(\pm)}_{pp',0} &= -(u_{mp}, w_{mp})^{-1}(u_{mp}, [I - M^{(0)}_{mp}T^{(0)}_{mp}]\eta^{(\pm)}_{pp'}w_{mp'}) \\ &\qquad \mp\tfrac{1}{2}(x_3 u_{mp}, T^{(0)}_{mp}\eta^{(\pm)}_{pp'}w_{mp'}) \end{aligned} \tag{2.20}$$

where x_3 is a multiplication operator by x_3 and

$$\eta^{(\pm)}_{pp'}(x_3) = \exp\left\{\pm\mathrm{i}x_3\sqrt{\epsilon_{mp} - \epsilon_{mp'}}\right\}.$$

(ii) $p > p'$:

$$\begin{aligned} t^{(\pm)}_{p'p}(E) &= \sum_{j=1}^{l-1}(\mathrm{i}k_{mp}(E))^j t^{(\pm)}_{p'p,l} + \mathrm{o}(k^{l-1}_{mp}) \\ r^{(\pm)}_{p'p}(E) &= \sum_{j=1}^{l-1}(\mathrm{i}k_{mp}(E))^j r^{(\pm)}_{p'p,j} + \mathrm{o}(k^{l-1}_{mp}) \end{aligned} \tag{2.21}$$

with the leading terms

$$t^{(\pm)}_{p'p,1} = \mathrm{i}(\epsilon_{mp} - \epsilon_{mp'})^{-1/2}\Big[(u_{mp}, w_{mp})^{-1}(\eta^{(\pm)}_{pp'}u_{mp'}, [I - T^{(0)}_{mp}M^{(0)}_{mp}]w_{mp})$$

$$\mp\tfrac{1}{2}(\eta^{(\pm)}_{pp'}u_{mp'}, T^{(0)}_{mp}x_3 w_{mp})\Big]$$

$$r^{(\pm)}_{p'p,1} = \mathrm{i}(\epsilon_{mp} - \epsilon_{mp'})^{-1/2}\Big[(u_{mp}, w_{mp})^{-1}(\eta^{(\mp)}_{pp'}u_{mp'}, [I - T^{(0)}_{mp}M^{(0)}_{mp}]w_{mp})$$

$$\mp\tfrac{1}{2}(\eta^{(\mp)}_{pp'}u_{mp'}, T^{(0)}_{mp}x_3 w_{mp})\Big].$$

Thus, the threshold behaviour of the S-matrix elements depends on the type of transition. Case (ii) corresponds to transitions to lower Landau levels. Case (i) corresponds to the elastic transition ($p = p'$) and to the channels of excitation of Landau levels ($p > p'$). As $\eta^{(\pm)}_{pp} = 1$ and $Q_{mp}w_{mp} = 0$, for the elastic transition the leading terms (2.20) are

$$t^{(\pm)}_{pp,0} = 0 \qquad r^{(\pm)}_{pp,0} = -1. \tag{2.22}$$

This follows

$$\big[S_m(\epsilon_{mp} + 0)\big]_{pp} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

whereas in the inelastic channels with $p' < p$ the elements $\big[S_m(\epsilon_{mp} + 0)\big]_{p'p}$ vanish due to (2.21) and (2.16). From equations (2.10), (2.11), (2.13) and assumption 2.2 it follows that the elements $\big[S_m(E)\big]_{p_1p_2}$ with $p_1, p_2 < p$ are continuous when E goes through ϵ_{mp}. Therefore, calculating the determinant of $S_m(\epsilon_{mp} + 0)$ yields the following result.

Corollary 2.6. For all $p \in \mathbb{N}_0 \setminus \{0\}$

$$\det S_m(\epsilon_{mp} + 0) = -\det S_m(\epsilon_{mp} - 0)$$

and $\det S_m(\epsilon_{m0} + 0) = -1$.

We now present results concerning the resolvent operators

$$R_m(z) = (H_m - z)^{-1} \qquad R_{0m}(z) = (H_{0m} - z)^{-1}$$

and their difference

$$\tilde{R}_m(z) = R_m(z) - R_{0m}(z). \tag{2.23}$$

We introduce the notation

$$z \to \infty \setminus \Gamma_m \equiv \{z \to \infty : \mathrm{dist}\,(z, \Gamma_m) \geqslant \delta > 0\} \tag{2.24}$$

to denote a limit when $z \to \infty$ out of a small neighbourhood of the set of all thresholds.

Theorem 2.7. The operator $\tilde{R}_m(z)$ is trace class for $\operatorname{Im} z \neq 0$. Its trace $\operatorname{tr} \tilde{R}_m(z)$ is analytic in z on the open cut plane $\Pi_m = \mathbb{C} \setminus [\epsilon_{m0}, \infty)$ and has simple poles in the points of discrete spectrum of H_m. Moreover, the following asymptotic expansions hold true:
(i) When $z \to \epsilon_{mp}$

$$\operatorname{tr} \tilde{R}_m(z) = \frac{1}{2\left(z - \epsilon_{mp}\right)} + \sum_{j=-1}^{l-4} (\mathrm{i}k_{mp}(z))^j \Delta_j + \mathrm{o}(|k_{mp}|^{l-4}) \tag{2.25}$$

where l is from condition (2.3) and $k_{mp}(z)$ is defined in (2.19).
(ii) When $z \to \infty \setminus \Gamma_m$

$$\operatorname{tr} \tilde{R}_m(z) = \frac{(-1)^{|m|}}{4\pi z} \int_{\mathbb{R}^2_+} \mathrm{d}\rho \, \mathrm{d}x_3 \; V(\rho, x_3)[1 + \mathrm{o}(1)]$$

uniformly in $\arg z$.

As $\tilde{R}_m(z)$ is trace class, its trace can be expressed through the Krein's spectral shift function $\xi_m(E)$ [36] (see also [29]) as

$$\operatorname{tr} \tilde{R}_m(z) = -\int_{-\infty}^{\infty} (E - z)^{-2} \xi_m(E) \, \mathrm{d}E. \tag{2.26}$$

Due to the abstract Birman–Krein theorem [37], this function is related to the S-matrix by

$$\det S_m(E) = \exp\{-2\pi i \xi_m(E)\} \tag{2.27}$$

a.e. in $E \in [\epsilon_{m0}, \infty)$. The l.h.s. is continuously differentiable on every interval $(\epsilon_{mp}, \epsilon_{mp+1})$ due to theorem 2.1. Therefore, the same holds true for the function $\xi_m(E)$. Taking the limit $\operatorname{Im} z \to 0$ of the imaginary part of (2.26) yields the trace relation for the operator H_m:

$$\operatorname{Im} \operatorname{tr} \tilde{R}_m(E \pm \mathrm{i}0) = \pm \frac{1}{2\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}E} \log \det S_m(E) \tag{2.28}$$

for $E \in [\epsilon_{m0}, \infty) \setminus \Gamma_m$. This relation underlies the following Levinson formula.

Theorem 2.8. The Levinson formula for the operator (2.1):

$$\pi(N_m - \tfrac{1}{2}) = \frac{1}{2\mathrm{i}} \log \det S_m(\epsilon_{m0}) - \lim_{E \to \infty \setminus \Gamma_m} \frac{1}{2\mathrm{i}} \log \det S_m(E) - \tfrac{1}{4}(-1)^{|m|} \int_{\mathbb{R}^2_+} \mathrm{d}\rho \, \mathrm{d}x_3 \; V(\rho, x_3) \tag{2.29}$$

where N_m is the number of eigenvalues of H_m counting multiplicity.

Note that due to corollary 2.6, $(2\pi\mathrm{i})^{-1} \log \det S_m(\epsilon_{m0})$ is integer plus one half. This means that

$$I = \frac{1}{2\mathrm{i}\pi} \lim_{E \to \infty \setminus \Gamma_m} \log \det S_m(E) - \frac{(-1)^{|m|}}{4\pi} \int_{\mathbb{R}^2_+} \mathrm{d}\rho \, \mathrm{d}x_3 \; V(\rho, x_3)$$

is integer. Taking the branch of the logarithm in (2.29) so that $I = 0$ reduces the Levinson formula to

$$\pi(N_m - 1/2) = \frac{1}{2\mathrm{i}} \log \det S_m(\epsilon_{m0}).$$

We now proceed to prove the results described above.

3. The S-matrix

In this section we prove theorems 2.3 and 2.4. We begin with the following technical result concerning the functions (2.17).

Lemma 3.1. Let the condition (2.3) hold true for some $l \geqslant 0$. Then the sum

$$S = \sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} \frac{(u_{mp}, (1+|x_3|)^{2l} u_{mp'})^2}{\sqrt{\epsilon_{mp}}\sqrt{\epsilon_{mp'}}} \tag{3.1}$$

converges.

Proof. Let

$$v_l(\rho) = \int_{\mathbb{R}} \mathrm{d}x_3 \, (1+|x_3|)^{2l} |V(\rho, x_3)|.$$

Then (3.1) can be rewritten as

$$S = \sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} \frac{(\phi_{mp}, v_l \phi_{mp'})^2}{\sqrt{\epsilon_{mp}}\sqrt{\epsilon_{mp'}}} = \mathrm{tr}\left(v_l (h_m^{\mathrm{osc}} + Bm)^{-1/2}\right)^2$$

where h_m^{osc} is defined in (2.2). Thus, it suffices to show that $v_l(h_m^{\mathrm{osc}} + Bm)^{-1/2}$ is Hilbert–Schmidt that implies that its square is trace class.

To this end consider the norm

$$\|v_l(h_m^{\mathrm{osc}} + Bm)^{-1/2}\|_2^2 = \int_0^{\infty} \rho \mathrm{d}\rho \, v_l^2(\rho) r_m^{\mathrm{osc}}(\rho, \rho, -Bm) \tag{3.2}$$

where $r_m^{\mathrm{osc}}(z)$ is the resolvent operator of the Hamiltonian h_m^{osc} with the kernel

$$r_m^{\mathrm{osc}}(\rho, \rho'; z) = \frac{1}{\sqrt{\rho\rho'}} \frac{\phi_1^{(m)}(\rho_<, z)\phi_2^{(m)}(\rho_>, z)}{\mathcal{W}_m(z)} \tag{3.3}$$

expressed through the Whittaker functions [30]

$$\phi_1^{(m)}(\rho, z) = r^{-1/4} M_{\nu\mu}(r)$$

$$\phi_2^{(m)}(\rho, z) = r^{-1/4} W_{\nu\mu}(r)$$

$$\mathcal{W}_m(z) = \frac{\sqrt{2B}|m|!}{\Gamma\left(-\frac{z}{2B} + \frac{|m|+1}{2}\right)}$$

where $r = B\rho^2/2$, $\nu = z/(2B)$, and $\mu = |m|/2$. Equation (3.2) can be rewritten as

$$\|v_l(h_m^{\mathrm{osc}} + Bm)^{-1/2}\|_2^2 = \mathcal{W}_m^{-1}(-Bm) \int_0^{\infty} \mathrm{d}\rho \, v_l^2(\rho) \phi_1^{(m)}(\rho, -Bm) \phi_2^{(m)}(\rho, -Bm).$$

By making use of the bound

$$|\phi_1^{(m)}(\rho, -Bm)\phi_2^{(m)}(\rho, -Bm)| \leqslant C_m \mathcal{W}_m(-Bm)\, \rho \chi_m(\rho) \tag{3.4}$$

with

$$\chi_m(\rho) = \begin{cases} (1+\rho^2)^{-1} & m \neq 0 \\ \log\rho[(1+\rho)(1+\rho\log\rho)]^{-1} & m = 0 \end{cases}$$

and appropriate constants C_m, one gets

$$\|v_l(h_m^{\mathrm{osc}} + Bm)^{-1/2}\|_2^2 \leqslant C_m \int_0^{\infty} \rho \mathrm{d}\rho \, v_l^2(\rho) \chi_m(\rho).$$

The last integral converges if potential satisfies (2.3) with $l \geqslant 0$.

We now proceed to study the operator $A_m(z)$ (2.12). The resolvent $R_{0m}(z)$ of the Hamiltonian H_{0m} can be written as

$$R_{0m}(z) = \sum_{p=0}^{\infty} P_{mp} r_0(z - \epsilon_{mp}) \tag{3.5}$$

where P_{mp} stands for the eigenprojector of the Hamiltonian h_m^{osc} (2.2)

$$P_{mp} = \langle \phi_{mp}, \cdot \rangle \phi_{mp} \tag{3.6}$$

and $r_0(z)$ is the resolvent operator of $h_0 = -\partial_{x_3}^2$ with the integral kernel

$$r_0(x_3, x_3'; z) = \frac{\mathrm{i}}{2\sqrt{z}} \exp\left(\mathrm{i}\sqrt{z}|x_3 - x_3'|\right) \qquad \operatorname{Im}\sqrt{z} > 0. \tag{3.7}$$

This representation shows that $A_m(z)$ has threshold singularities of the form

$$A_m(z) = \frac{\mathrm{i}}{2} \sum_{p=0}^{\infty} \frac{\mathcal{P}_{mp}}{k_{mp}(z)} (u_{mp}, w_{mp}) + M_m(z) \tag{3.8}$$

where the constituents of the sum are defined in (2.18), (2.19) and smooth operator $M_m(z)$ is given by

$$M_m(z) = \sum_{p=0}^{\infty} |V|^{1/2} P_{mp} \tilde{r}_0(z - \epsilon_{mp}) V^{1/2} \tag{3.9}$$

with

$$\tilde{r}_0(x_3, x_3'; z) = -\frac{1}{\sqrt{z}} \exp\left(\frac{\mathrm{i}\sqrt{z}}{2}|x_3 - x_3'|\right) \sin\left(\frac{\sqrt{z}}{2}|x_3 - x_3'|\right) \qquad \operatorname{Im}\sqrt{z} > 0. \tag{3.10}$$

Theorem 3.2. The operator $M_m(z)$ is Hilbert–Schmidt for all $z \in \mathbb{C}$, holomorphic on the open cut plane $\Pi_m = \mathbb{C} \setminus [\epsilon_{m0}, \infty)$ and norm continuous in $\overline{\Pi}_m$. The same holds true for $A_m(z)$ and $\partial_z A_m(z)$ except for the points of the set Γ_m.

Proof. Consider the norm

$$\|M_m(z)\|_2^2 \leqslant \sum_{p,p'=0}^{\infty} \int \mathrm{d}x_3 \int \mathrm{d}x_3' \, V_{pp'}(x_3) V_{pp'}(x_3') \times |\tilde{r}_0(x_3, x_3'; z - \epsilon_{mp}) \tilde{r}_0(x_3, x_3'; \bar{z} - \epsilon_{mp'})|$$

where

$$V_{pp'}(x_3) = \langle \phi_{mp}, |V(\cdot, x_3)| \phi_{mp'} \rangle = \langle u_{mp}(\cdot, x_3), u_{mp'}(\cdot, x_3) \rangle.$$

Substituting explicit expressions (3.10) for the resolvent kernels and making use of the bounds

$$\left|\mathrm{e}^{\mathrm{i}kx/2} \sin(kx/2)\right| \leqslant c \frac{|k|x}{1 + |k|x} \qquad x \geqslant 0$$

and

$$\frac{|x-y|^2}{(1 + |k||x-y|)(1 + |k'||x-y|)} \leqslant \frac{(1+|x|)^2 (1+|y|)^2}{(1+|k|)(1+|k'|)}$$

leads one to

$$\|M_m(z)\|_2^2 \leqslant c^2 \sum_{p,p'=0}^{\infty} \frac{(u_{mp}, (1+|x_3|)^2 u_{mp'})^2}{(1+|k_{mp}|)(1+|k_{mp'}|)}.$$

The sum converges due to lemma 3.1, so that $M_m(z)$ is Hilbert–Schmidt for all $z \in \mathbb{C}$.

We now prove that the sum from (3.8) converges in the Hilbert–Scmidt norm for all $z \overline{\in} \Gamma_m$. This directly follows from lemma 3.1 and the bound

$$\left\| \sum_{p=0}^{\infty} (u_{mp}, w_{mp}) \frac{\mathcal{P}_{mp}}{k_{mp}} \right\|_2^2 = \sum_{p,p'=0}^{\infty} \frac{(u_{mp}, u_{mp'})^2}{\sqrt{z - \epsilon_{mp}}\sqrt{\overline{z} - \epsilon_{mp'}}}$$

$$\leqslant \left[\sup_p \left\{ \sqrt{\frac{\epsilon_{mp}}{|z - \epsilon_{mp}|}} \right\} \right]^2 \sum_{p,p'=0}^{\infty} \frac{(u_{mp}, u_{mp'})^2}{\sqrt{\epsilon_{mp}}\sqrt{\epsilon_{mp'}}}$$

so that $A_m(z)$ is also Hilbert–Schmidt for $z \in \mathbb{C} \setminus \Gamma_m$.

The boundedness of the norm $\|\partial_z A_m(z)\|_2$ for $z \overline{\in} \Gamma_m$ can be proved as above by making use of the bound

$$\left| \frac{\partial}{\partial z} r_0(x_3, x_3'; z - \epsilon_{mp}) \right| \leqslant \frac{|x_3 - x_3'|}{4|z - \epsilon_{mp}|} + \frac{1}{4|z - \epsilon_{mp}|^{3/2}} \qquad (3.11)$$

that follows from (3.7). To prove the continuity of $\partial_z A_m(z)$ one can easily show that

$$\left\| \frac{\partial}{\partial z} A_m(z) - \frac{\partial}{\partial z} A_m(z + \delta) \right\|_2 \to 0$$

when $\delta \to 0$ for $z \overline{\in} \Gamma_m$.

Theorem 3.2 enables one to apply the analytic Fredholm theorem [32] to invert the operator $I + A_m(z)$. Denote yb $\mathcal{E}_m \subset \mathbb{R}$ the set of z for which the homogeneous equation $\phi + A_m(z)\phi = 0$ has a non-trivial solution. The set $\mathcal{E}_m \cap (a, b)$ is a closed set of null Lebesgue measure for arbitrary interval $(a, b) \cap \Gamma_m = \emptyset$. Let $\mathcal{E}_m^{(-)} = \mathcal{E}_m \cap (-\infty, \epsilon_{m0})$ and $\mathcal{E}_m^{(+)} = \mathcal{E}_m \setminus \mathcal{E}_m^{(-)}$. The operator

$$T_m(z) = [I + A_m(z)]^{-1} \qquad (3.12)$$

exists for z on the open cut plane $\Pi_m = \mathbb{C} \setminus [\epsilon_{m0}, \infty)$. It is meromorphic in Π_m with finite-dimensional residues at poles $z \in \mathcal{E}_m^{(-)}$ and norm continuously differentiable in $\overline{\Pi}_m \setminus \{\Gamma_m \cup \mathcal{E}_m^{(+)}\}$. The set $\mathcal{E}_m^{(-)}$ consists from the eigenvalues of H_m.

Due to theorems 4.5 and 4.6 of [2], the singular continuous spectrum of H_m is empty. Also, from the general result of Froese *et al* [33] it follows that H_m has no embedded eigenvalues. Thus, $\mathcal{E}_m^{(+)}$ may only consist of some scattering thresholds ϵ_{mp} (even if H_m has no eigenvalues at thresholds, the limit of $T_m(z)$ as z goes to some ϵ_{mp} may not exist). In this case the corresponding thresholds are exceptional points. We do not consider any case that is excluded by assumption 2.2. Thus, in our case $\mathcal{E}_m^{(+)} = \emptyset$.

Proof of theorem 2.3. Equation (2.13) is standard. It is based on the following representation of the scattering operator S_m (see, e.g., [31]):

$$S_m - I = s - \lim_{\delta_1 \downarrow 0} s - \lim_{\delta_2 \downarrow 0} \left\{ - \int_{-\infty}^{\infty} [R_{0m}(\lambda - \mathrm{i}\delta_1) - R_{0m}(\lambda + \mathrm{i}\delta_1)] \right.$$

$$\left. \times (V + V R_m(\lambda + \mathrm{i}\delta_2) V) \, E_{0m}(\mathrm{d}\lambda) \right\}$$

with $E_{0m}(\cdot)$ being the spectral measure of H_{0m}. To show that $S_m(E)$ is continuously differentiable, it suffices to prove this for the operator $F_{V^{1/2}}(E)$ (2.10). That follows from the bound

$$\left\| \frac{\partial F_{V^{1/2}}(E)}{\partial E} \right\|_2^2 \leqslant \frac{1}{32} \sum_{p=0}^{n(E)} \frac{(w_{mp}, w_{mp})}{k_{mp}^5(E)} + \frac{1}{8} \sum_{p=0}^{n(E)} \frac{(w_{mp}, x_3 w_{mp})}{k_{mp}^4(E)} + \frac{1}{8} \sum_{p=0}^{n(E)} \frac{(w_{mp}, x_3^2 w_{mp})}{k_{mp}^3(E)}.$$

Proof of theorem 2.4. Due to theorem 3.2, equation (2.14) has a unique solution for $E \in (\epsilon_{m0}, \infty) \setminus \Gamma_m$. Then the function $\varphi_m^{(\pm)}(\cdot; p, E)$ is correctly defined for $R_{0m}(E \pm 0)V^{1/2}$ is bounded as a map from $L^2(\mathbb{R}^2_+)$ to $L^\infty(\mathbb{R}^2_+)$. It remains for us to prove the asymptotics (2.15). Due to (2.14) the function $\varphi_m^{(\pm)}(\cdot; p, E)$ is represented as follows:

$$\varphi_m^{(\pm)}(\cdot; p, E) = f^{(\pm)}(\cdot; p, E) + g^{(\pm)}(\cdot; p, E)$$

with

$$f^{(\pm)}(\cdot; p, E) = \sqrt{2\pi} \left[I - \sum_{p'=0}^{n(E)} G_{p'}(E) T_m(E + \mathrm{i}0) |V|^{1/2} \right] \mathcal{F}_m(p, \pm k_{mp}(E); \cdot) \tag{3.13}$$

where $T_m(z)$ is defined by (3.12) and

$$G_p(E) = P_{mp} r_0(E - \epsilon_{mp} + \mathrm{i}0) V^{1/2}.$$

The function $g^{(\pm)}$ is given by the same equation where the sum is evaluated over $p' > n(E)$. As the operator $\sum_{p'>n(E)} G_{p'}(E)$ is bounded, the function $g^{(\pm)}$ is squared integrable and does not contribute into the asymptotics (2.15).

Upon using the obvious asymptotics of the resolvent kernel r_0 (3.7) in (3.13) one gets (2.15) with the coefficients given by

$$\begin{aligned} t_{p'p}^{(\pm)}(E) &= \delta_{p'p} - \frac{\mathrm{i}}{2k_{mp'}(E)} \left(\overline{\Phi_{mp'}^{(\mp)}(E)}, T_m(E + \mathrm{i}0) \Psi_{mp}^{(\pm)}(E) \right) \\ r_{p'p}^{(\pm)}(E) &= -\frac{\mathrm{i}}{2k_{mp'}(E)} \left(\overline{\Phi_{mp'}^{(\pm)}(E)}, T_m(E + \mathrm{i}0) \Psi_{mp}^{(\pm)}(E) \right) \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \Phi_{mp}^{(\pm)}(\rho, x_3; E) &= \exp\left(\pm \mathrm{i} x_3 k_{mp}(E)\right) u_{mp}(\rho, x_3) \\ \Psi_{mp}^{(\pm)}(\rho, x_3; E) &= \exp\left(\pm \mathrm{i} x_3 k_{mp}(E)\right) w_{mp}(\rho, x_3). \end{aligned} \tag{3.15}$$

Comparing these formulae with (2.13) yields (2.16).

4. Near-threshold and high-energy expansions

In this section we prove theorems 2.5 and 2.7. We begin with the following auxiliary result concerning the operator (3.9). Recall that we assume the condition (2.3) to be satisfied for some $l \geqslant 2$.

Lemma 4.1. In the limit $z \to \epsilon_{mp}$ the operator $M_{mp}(z)$ has the asymptotics

$$M_{mp}(z) = \sum_{j=0}^{l-1} (\mathrm{i}k_{mp}(z))^j M_{mp}^{(j)} + o(k_{mp}^{l-1})$$

which are valid in the Hilbert–Schmidt norm.

Proof. It suffices to show that the operator $\left(\partial^j / \partial k_{mp}^j M_{mp}\right)\left(z = \epsilon_{mp}\right)$ is Hilbert–Schmidt for all $j \leqslant l - 1$. This can be done by making use of (3.11), in a manner similar to the proof of theorem 3.2.

The next result concerns the threshold behaviour of the operator (3.12). By assumption 2.2, $T_m(\epsilon_{mp})$ is correctly defined.

Lemma 4.2. Near the threshold ϵ_{mp} the operator $T_m(z)$ can be expanded as follows:

$$T_m(z) = \sum_{j=0}^{l-1} (\mathrm{i}k_{mp}(z))^j T_{mp}^{(j)} + \mathrm{o}(k_{mp}^{l-1}) \tag{4.1}$$

which is valid in the operator norm. The leading terms are given by

$$T_{mp}^{(0)} = [I + \mathcal{Q}_{mp} M_{mp}^{(0)} \mathcal{Q}_{mp}]^{-1} \mathcal{Q}_{mp}$$

$$T_{mp}^{(1)} = -2a\mathcal{P}_{mp} - T_{mp}^{(0)} \left(M_{mp}^{(1)} + 2aM_{mp}^{(0)}\mathcal{P}_{mp}M_{mp}^{(0)}\right) T_{mp}^{(0)} + 2a\left(T_{mp}^{(0)}M_{mp}^{(0)}\mathcal{P}_{mp} + \mathcal{P}_{mp}M_{mp}^{(0)}T_{mp}^{(0)}\right)$$

where $a = (u_{mp}, w_{mp})^{-1}$.

The proof can be carried out by adjusting the technique of [7].

Proof of theorem 2.5. For $p' \leqslant p$ the functions $\Phi_{mp'}^{(\pm)}$ and $\Psi_{mp'}^{(\pm)}$ (3.15) can be expanded near ϵ_{mp} as follows:

$$\begin{aligned}
\Phi_{mp'}^{(\pm)}(E) &= \sum_{j=0}^{l-1} (\mathrm{i}k_{mp})^j \eta_{pp'}^{(\pm)} \Phi_{mp',j}^{(\pm)} + \mathrm{o}(k_{mp}^{l-1}) \\
\Psi_{mp'}^{(\pm)}(E) &= \sum_{j=0}^{l-1} (\mathrm{i}k_{mp})^j \eta_{pp'}^{(\pm)} \Psi_{mp',j}^{(\pm)} + \mathrm{o}(k_{mp}^{l-1})
\end{aligned} \tag{4.2}$$

which is valid in L^2-norm. The leading coefficients are given by

$$\Phi_{mp',0}^{(\pm)} = u_{mp'} \qquad \Psi_{mp',0}^{(\pm)} = w_{mp'}$$

$$\Phi_{mp',1}^{(\pm)}(\rho, x_3) = \begin{cases} 0 & p' < p \\ \pm x_3 u_{mp}(\rho, x_3) & p' = p \end{cases}$$

$$\Psi_{mp',1}^{(\pm)}(\rho, x_3) = \begin{cases} 0 & p' < p \\ \pm x_3 w_{mp}(\rho, x_3) & p' = p. \end{cases}$$

Substituting (4.2) and (4.1) in (3.14) and using $\mathcal{Q}_{mp} w_{mp} = 0$, $\mathcal{Q}_{mp}^* u_{mp} = 0$ leads one to the statement of the theorem.

Lemma 4.3. The operators $R_{0m}(z)V^{1/2}$ and $|V|^{1/2}R_{0m}(z)$ are Hilbert–Schmidt for all $z \in \Pi_m$.

Proof. Consider, for instance, the operator $|V|^{1/2}R_{0m}(z)$. Represent it as follows

$$|V|^{1/2}R_{0m}(z) = |V|^{1/2}D^{-1}DR_{0m}(z) \tag{4.3}$$

where $D = (h_m^{\mathrm{osc}} + Bm)^{1/2} r_0^{-1/2}(-1)$, the operators h_m^{osc} and $r_0(z)$ being defined in (2.1), (3.7). The operator $DR_{0m}(z)$ is bounded for $\operatorname{Im} z \neq 0$ due to

$$\sup_{p,k_3} \left\{ \frac{(1+k_3^2)^{1/2}(\epsilon_{mp} + Bm)^{1/2}}{|k_3^2 + \epsilon_{mp} - z|} \right\} < \infty.$$

Consider the Hilbert–Schmidt norm

$$\| \, |V|^{1/2}D^{-1} \, \|_2^2 = \int_0^\infty \rho \mathrm{d}\rho \; r_m^{\mathrm{osc}}(\rho, \rho, -Bm) v(\rho)$$

where

$$v(\rho) = \int_{-\infty}^\infty \mathrm{d}x_3 \; |V(\rho, x_3)| r_0(x_3, x_3; -1) = \frac{1}{2} \int_{-\infty}^\infty \mathrm{d}x_3 \; |V(\rho, x_3)|.$$

By making use of Eqs. (3.3) and (3.4) one gets

$$\| \, |V|^{1/2} D^{-1} \, \|_2^2 \leqslant \tfrac{1}{2} C_m \int_{\mathbb{R}^2_+} \rho \mathrm{d}\rho \; \mathrm{d}x_3 \; |V(\rho, x_3)| \chi_m(\rho)$$

so that the product (4.3) is Hilbert–Schmidt. The operator $R_{0m}(z)V^{1/2}$ can be considered in the same way.

We now proceed to study the resolvent's difference (2.23). Consider the Born expansion for its trace

$$\operatorname{tr} \tilde{R}_m(z) = \sum_{n=1}^{\infty} f_n(z) \qquad f_n(z) = \operatorname{tr} \left[(-R_{0m}(z)V)^n R_{0m}(z) \right]. \tag{4.4}$$

The following result shows that this series is an asymptotic expansion as $z \to \infty$.

Lemma 4.4. Let conditions (2.3) hold true with $\mu_0 = 1$. Then in the limit $z \to \infty \setminus \Gamma_m$ (2.24) the functions (4.4) have the following asymptotics uniformly in $\arg z$:

$$f_n(z) = c_n z^{-n} + \mathrm{o}\left(z^{-n}\right) \tag{4.5}$$

where

$$c_n = (-1)^{|m|n} \frac{\mathrm{e}^{\mathrm{i}\pi(1-n)/4}}{4\pi n} \int_0^{\infty} \mathrm{d}\rho \int_{-\infty}^{\infty} \mathrm{d}x \; V^n(\rho, x).$$

Proof. We give the proof for the case $\operatorname{Im} z > 0$. Consider equation (3.5) for the operator $R_{0m}(z)$. It can be rewritten as

$$R_{0m}(z) = \mathrm{i} \sum_{p=0}^{\infty} P_{mp} \int_0^{\infty} \mathrm{d}t \; \mathrm{e}^{\mathrm{i}t(z-\epsilon_{mp})} \mathrm{e}^{-\mathrm{i}th_0}$$

where the last exponent is the free propagator

$$\exp\{-\mathrm{i}th_0\}(x, x') = \frac{1}{2\sqrt{\pi t}} \exp\left\{-\mathrm{i}\frac{(x - x')^2}{4t}\right\} \qquad t > 0.$$

Substituting this in (4.4) yields

$$\begin{aligned} f_n(z) = -\left(\frac{-\mathrm{i}}{2\sqrt{\pi}}\right)^{n+1} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\mathrm{d}t_0 \, \mathrm{d}t_1 \cdots \mathrm{d}t_n}{\sqrt{t_0 t_1 \cdots t_n}} \exp\{\mathrm{i}z\,(t_0 + t_1 + \cdots + t_n)\} \\ \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}x_0 \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp\left\{-\mathrm{i} \sum_{j=0}^{n} \frac{\left(x_j - x_{j+1}\right)^2}{4t_j}\right\} \\ \times F\left(t_1, \ldots, t_{n-1}, t_n + t_0; x_1, \ldots, x_n\right) \end{aligned} \tag{4.6}$$

where $x_{n+1} \equiv x_0$,

$$F(t_1, \ldots, t_n; x_1, \ldots, x_n) = \sum_{p_1=0}^{\infty} \cdots \sum_{p_n=0}^{\infty} \exp\left\{-\mathrm{i} \sum_{j=1}^{n} t_j \epsilon_{mp_j}\right\} \prod_{j=1}^{n} V_{p_j p_{j+1}}(x_j) \tag{4.7}$$

with $p_{n+1} \equiv p_1$ and

$$V_{pp'}(x) = \langle \phi_{mp}, V(\cdot, x)\phi_{mp'} \rangle. \tag{4.8}$$

The integral over x_0 in (4.6) involves only an exponent of a quadratic polynomial in x_0 and is evaluated explicitly. After that the dependence of the resulting integrand on t_0, t_n is only through the sum t_0+t_n. Passing to new variables $\tau_\pm = t_0 \pm t_n$, calculating the integral over τ_-, and denoting $\tau_+ = t_n$ yields

$$f_n(z) = e^{i\pi/4}\left(\frac{-i}{2\sqrt{\pi}}\right)^n \int_0^\infty \cdots \int_0^\infty dt_1 \cdots dt_n \sqrt{\frac{t_n}{t_1\cdots t_{n-1}}} \exp\{izT\}$$
$$\times \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty dx_1 \cdots dx_n \exp\left\{-i\sum_{j=1}^n \frac{(x_j - x_{j+1})^2}{4t_j}\right\}$$
$$\times F(t_1,\ldots,t_n;x_1,\ldots,x_n) \tag{4.9}$$

where $x_{n+1} \equiv x_1$ and $T = t_1 + J \cdots + t_n$.

In (4.9) the z-dependence of integrand is factorized in the first exponent. That is why the asymptotics at $z \to \infty$ are determined by a vicinity of the point $t_1 = \cdots = t_n = 0$. First, let us evaluate the asymptotics of F near this point. To this end we rewrite (4.7) in an integral form. Substituting in (4.8) the explicit expressions (2.7) for the function φ_{mp} and making use of the addition formula for the Legendre polynomials [34]

$$\sum_{p=0}^\infty c_{mp}^2 L_p^{|m|}(r) L_p^{|m|}(r')\zeta^p = B\frac{(rr'\zeta)^{-|m|/2}}{1-\zeta}\exp\left\{-\zeta\frac{r+r'}{1-\zeta}\right\} I_{|m|}\left(2\frac{\sqrt{rr'\zeta}}{1-\zeta}\right)$$

where $I_{|m|}$ stands for the modified Bessel function, leads one to the representation

$$F(t_1,\ldots,t_n;x_1,\ldots,x_n) = i^{-n}\exp\left\{-iBmT + i\pi\frac{|m|n}{2}\right\}$$
$$\times \int_0^\infty \cdots \int_0^\infty \rho_1 d\rho_1 \cdots \rho_n d\rho_n\, V(\rho_1,x_1)\cdots V(\rho_n,x_n)$$
$$\times \prod_{j=1}^n \lambda_j \exp\left\{i\frac{\lambda_j}{2}\left(\rho_j^2+\rho_{j+1}^2\right)\cos Bt_j\right\} J_{|m|}\left(\lambda_j\rho_j\rho_{j+1}\right) \tag{4.10}$$

where $\rho_{n+1} \equiv \rho_1$ and $\lambda_j = B/\left(2\sin Bt_j\right)$.

In the limit $t_j \to 0$ ($\lambda_j \to \infty$) one can replace the Bessel functions by their asymptotics

$$J_m(\lambda) = (2\pi\lambda)^{-1/2}\sum_\pm \exp\left(\pm i\lambda \mp i\pi m/2 \mp i\pi/4\right) + O\left(\lambda^{-3/2}\right). \tag{4.11}$$

Upon substituting this in (4.10) it is readily seen that the leading order of the asymptotics is due to the terms of (4.11) with the minus sign, so that

$$F(t_1,\ldots,t_n;x_1,\ldots,x_n) \sim (-1)^{|m|n}\left(\frac{-i}{2\pi}\right)^{n/2}$$
$$\times \int_0^\infty \cdots \int_0^\infty d\rho_1 \cdots d\rho_n\, V(\rho_1,x_1J)\cdots V(\rho_n,x_nJ)$$
$$\times \prod_{j=1}^n \sqrt{\lambda_j}\exp\left\{i\frac{\lambda_j}{2}\left(\rho_j-\rho_{j+1}\right)^2\right\}. \tag{4.12}$$

This integral is evaluated by making use of the asymptotics ($\lambda_j \to \infty$)

$$\int_0^\infty d\rho\ f(\rho)\exp\left\{i\frac{\lambda_1}{2}(\rho_1-\rho)^2 + i\frac{\lambda_2}{2}(\rho-\rho_2)^2\right\}$$
$$\sim e^{i\pi/4}\sqrt{\frac{2\pi}{\lambda_1+\lambda_2}}\exp\left\{i\frac{\lambda_1\lambda_2}{2(\lambda_1+\lambda_2J)}(\rho_1-\rho_2)^2\right\} fJ\left(\frac{\lambda_1\rho_1+\lambda_2\rho_2}{\lambda_1+\lambda_2}J\right) \tag{4.13}$$

which follows from the Erdelyi lemma [35]. Applying it repeatedly to the integrals of (4.12) over $\rho_2, \ldots, \rho_n$ yields

$$F(t_1, \ldots, t_n; x_1, \ldots, x_n) \sim \tfrac{1}{2}(-1)^{|m|n} e^{-i\pi/4} (\pi T)^{-1/2} \int_0^\infty d\rho\, V(\rho, x_1 J) \cdots V(\rho, x_n J)$$

which describes the asymptotics of F as $t_j \to 0$.

Upon substituting these asymptotics in (4.9) for $f_n(z)$ the integral over $x_1, \ldots, x_n$ takes a form like (4.12). Its asymptotics as $t_j \to 0$ can be evaluated by means of (4.13) to yield

$$f_n(z) \sim \frac{1}{4\pi}(-1)^{(|m|+1)n} e^{i\pi(n+1)/4} \int_0^\infty d\rho \int_{-\infty}^\infty dx\, V^n(\rho, x)$$
$$\times \int_0^\infty \cdots \int_0^\infty dt_1 \cdots dt_n \left(\frac{t_n}{T} J\right) e^{iTz}.$$

The last integral is calculated explicitly by passing to new variables

$$T_k = \sum_{j=1}^k t_j \qquad k = 1, 2, \ldots, n.$$

This yields the asymptotics (4.5).

Proof of theorem 2.7. From the relation

$$\tilde{R}_m(z) = -R_{0m}(z) V^{1/2} T_m(z) |V|^{1/2} R_{0m}(z)$$

and lemma 4.1, it follows that $\tilde{R}_m(z)$ is trace class for $\operatorname{Im} z \neq 0$. The last equation follows

$$\operatorname{tr} \tilde{R}_m(z) = -\operatorname{tr}\left[|V|^{1/2} R_{0m}^2(z) V^{1/2} T_m(z)\right] = -\operatorname{tr}\left[T_m(z) \frac{\partial}{\partial z} A_m(z)\right].$$

As z goes to ϵ_{mp}, by making use of decomposition (2.19) one can rewrite this in the form

$$\operatorname{tr} \tilde{R}_m(z) = \tfrac{1}{4}(ik_{mp}(z))^{-3}(u_{mp}, T_m(z) w_{mp}) - \frac{1}{2k_{mp}(z)} \operatorname{tr}\left[T_m(z) \frac{\partial}{\partial z} M_{mp}(z)\right].$$

Lemmas 4.1 and 4.2 follow statement (i) of the theorem. Lemma 4.4 follows statement (ii).

5. The Levinson formula

We begin with a result that extends corollary 2.6.

Lemma 5.1. For all $p = 1, 2, \ldots$

$$\log\det S_m\left(\epsilon_{mp} - 0\right) - \log\det S_m\left(\epsilon_{mp} + 0\right) = \pi i. \tag{5.1}$$

Proof. Due to theorem 2.7, the function $\operatorname{tr} \tilde{R}_m(z)$ has a simple pole in $z = \epsilon_{mp}$ with residue $\frac{1}{2}$. Integrating it anticlockwise along a small loop γ_p around ϵ_{mp} yields

$$\oint_{\gamma_p} dz\ \operatorname{tr} \tilde{R}_m(z) = \pi i.$$

Due to (2.26), the integral can be as well calculated through the spectral shift function to give

$$\oint_{\gamma_p} dz\ \operatorname{tr} \tilde{R}_m(z) = 2\pi i \lim_{\delta \to 0}\left[\xi_m\left(\epsilon_{mp} + \delta\right) - \xi_m\left(\epsilon_{mp} - \delta\right)\right].$$

Therefore, the spectral shift function has a jump in every threshold:

$$\xi_m\left(\epsilon_{mp} + 0\right) - \xi_m\left(\epsilon_{mp} - 0\right) = \tfrac{1}{2}.$$

The relation (2.27) between $\xi_m(E)$ and the S-matrix follows (5.1).

Proof of theorem 2.8. Let M_m be the number of eigenvalues of the Hamiltonian H_{0m}, μ_j be their multiplicities and $N_m = \sum_{j=1}^{M_m} \mu_j$. Define the function

$$F(z) = \operatorname{tr} \tilde{R}_m(z) - \frac{(-1)^{|m|}}{4\pi z} \int_{\mathbb{R}^2_+} d\rho \, dx_3 \; V(\rho, x_3)$$

where the last term is the leading order of the asymptotics of the trace as $z \to \infty \setminus \Gamma_m$. Theorem 2.7 follows that $F(z)$ is analytic on Π_m. It has simple poles with residues $\frac{1}{2}$ in the points ϵ_{mp}, M_m poles in the eigenvalues of H_m with residues $-\mu_j$, and a simple pole in the origin.

Let R be a large positive number lying between two thresholds $\epsilon_{mn}, \epsilon_{mn+1}$ with $n = n(R)$. Consider a contour C that goes from a point $\epsilon_{m0} - \delta$ along the upper cut $z = E + i0$ to the point $R + i0$, then anticlockwise along circle C_R of radius R to the point $R - i0$ and then along the lower cut $z = E - i0$ to the initial point $\epsilon_{m0} - \delta$. The integral of $F(z)$ along this contour is the sum of residues in the eigenvalues and in the origin:

$$\oint_C F(z) \, dz = -2\pi i N_m - \frac{i}{2}(-1)^{|m|} \int_{\mathbb{R}^2_+} d\rho \, dx_3 \; V(\rho, x_3) .$$

In the limit $R \to \infty \setminus \Gamma_m$ contribution of the circle C_R vanishes as $F(z) = o\left(z^{-1}\right)$ due to theorem 2.7. By making use of (2.25) and (2.28), the integrals along the upper and lower cuts are expressed through the S-matrix and the sum of residues in the thresholds. One gets

$$-\log\det S_m(\epsilon_{m0} + 0J) - \pi i + \log\det S_m(R) + \sum_{p=1}^{n(R)} \log\det S_m(E)\Big|_{\epsilon_{mp}-0}^{\epsilon_{mp}+0} - \pi i n(R)$$

$$= -2\pi i N_m - \frac{i}{2}(-1)^{|m|} \int_{\mathbb{R}^2_+} d\rho \, dx_3 \; V(\rho, x_3) .$$

Due to lemma 5.1, the jumps of the S-matrix in thresholds cancel the last term of the l.h.s. In the limit $R \to \infty \setminus \Gamma_m$ one gets (2.29).

Acknowledgment

It is painful to think that Professor Stanislav P Merkuriev is no longer among us, and that this paper is his last contribution to quantum scattering theory which owes remarkable results to him. His intuition, insight and support were invaluable for two other authors during our long-standing collaboration. We are deeply grateful to him.

References

[1] McDowell M R C and Zarcone M 1986 *Adv. At. Mol. Phys.* **21** 255
[2] Avron J, Herbst I and Simon B 1978 *Duke Math. J.* **45** 847
[3] Landau L D and Lifshitz E M 1977 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Pergamon)
[4] Newton R G 1982 *Scattering Theory of Waves and Particles* (Berlin: Springer)
[5] Bollé D and Gesztesy F 1984 *Phys. Rev.* A **30** 1279
[6] Kvitsinsky A A 1984 *Theor. Math. Phys.* **59** 629; 1985 *Theor. Math. Phys.* **65** 1123
[7] Bollé D, Gesztesy F and Wilk S E J 1985 *J. Operator Theory* **13** 3
[8] Bollé D, Gesztesy F and Klaus M 1987 *J. Math. Anal. Appl.* **122** 496
[9] Bollé D, Gesztesy F, Danneels C and Wilk S F J 1986 *Phys. Rev. Lett.* **56** 900
[10] Levinson N 1949 *Kgl. Danske Vidensk. Selskab. Mat.-Fys. Medd.* **25** 1
[11] Buslaev V S and Faddeev L D 1960 *Dokl. Akad. Nauk USSR* **132** 13 (in Russian)
[12] Buslaev V S 1962 *Sov. Phys.–Dokl.* **7** 295 (reprinted 1967 *Topics in Mathematical Physics* vol 1, ed M Sh Birman (New York: Consultant Bureau) p 69)

[13] Cheney M 1984 *J. Math. Phys.* **25** 1449
[14] Sahnovich L A 1966 *Izv. Akad. Nauk USSR Math. Ser.* **30** 1297 (in Russian)
[15] Yafaev D R 1972 *Theor. Math. Phys.* **11** 358
[16] Kvitsinsky A A 1986 *Theor. Math. Phys.* **68** 801; 1987 *Theor. Math. Phys.* **70** 72
[17] Klaus M 1988 *J. Math. Phys.* **29** 148
[18] Newton R G 1977 *J. Math. Phys.* **18** 1582
[19] Newton R G 1989 *Ann. Phys.* **194** 173
[20] Newton R G 1991 *J. Math. Phys.* **32** 551
[21] Buslaev V S and Merkuriev S P 1970 *Theor. Math. Phys.* **5** 1216
[22] Bollé D and Osborn T A 1982 *Phys. Rev.* A **26** 3062
[23] Buslaev V S and Chernenko V G 1987 *Vestnik LGU* No 2 16 (in Russian)
[24] Kvitsinsky A A and Merkuriev S P 1989 *Lett. Math. Phys.* **17** 307
[25] Kvitsinsky A A, Kostrykin V V and Merkuriev S P 1990 *Sov. J. Part. Nucl.* **21** 553
[26] Bollé D, Gesztesy F, Grosse H and Simon B 1987 *Lett. Math. Phys.* **13** 127
[27] Bollé D, Gesztesy F, Grosse H, Schweiger W and Simon B 1987 *J. Math. Phys.* **28** 1512
[28] Borisov N, Müller W and Schrader R 1988 *Commun. Math. Phys.* **114** 475
[29] Birman M Sh and Yafaev D R 1992 *Algebra i Analyz* **4** 1 (in Russian)
[30] Erdélyi A (ed) 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)
[31] Baumgärtel H and Wollenberg M 1983 *Mathematical Scattering Theory* (Berlin: Akademie)
[32] Reed M and Simon B 1979 *Methods of Modern Mathematical Physics III: Scattering Theory* (New York: Academic)
[33] Froese R, Herbst I, Hoffmann-Ostenhof M and Hoffmann-Ostenhof T 1982 *J. Anal. Math.* **41** 272
[34] Gradshteyn I S and Ryzhik I M 1965 *Tables of Integrals, Series and Products* (New York: Academic)
[35] Fedoryuk M V 1987 *Asymptotics: Integrals and Series* (Moscow: Nauka) (in Russian)
[36] Krein M G 1962 *Sov. Math.–Dokl.* **3** 707
[37] Birman M Sh and Krein M G 1962 *Sov. Math.–Dokl.* **3** 740